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Stability and branching of the relative equilibria of a three-link pendulum in a rapidly rotating frame of reference☆

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ABSTRACT

The relative equilibria of a plane three-link pendulum with a suspension rotating about a vertical axis in a gravitational force field are investigated. The pendulum is modelled as a system of three point masses, joined in tandem by massless non-elastic rods using cylindrical hinges. All the trivial equilibrium positions, their stability and branching are investigated. Attention is mainly paid to the non-trivial equilibrium positions, that is, to those equilibrium configurations of the pendulum for which not all the links are stretched out along the vertical axis. Questions of the existence, stability and the branching of these non-trivial equilibrium positions are investigated at fairly high angular velocities of rotation. The plane of the geometric parameters of the pendulum is subdivided into domains with a different number of non-trivial relative equilibria. Conditions are presented for each non-trivial equilibrium position which is found, and these are imposed on the system parameters for which an equilibrium position exists, their stability is investigated and a geometric description of the configuration of the pendulum is given.

The study of the dynamics of a body suspended on a string (rod) from a horizontal axis rotating about a vertical in a uniform gravitational field has its origin in the experimental investigations carried out under the supervision of Lavrent'ev. Theoretical investigations of the dynamics of a body (or several bodies) with a string drive have been developed by a number of authors.[1–9] The problem of the relative equilibrium of a heavy thread, rotating about the vertical, has also been investigated in different formulations.[10–13]

1. Formulation of the problem

Consider a three-link mathematical pendulum located in the Oxy plane which rotates at a constant angular velocity ω about the vertical Oy axis. The lengths of the links (massless rods) of the pendulum are denoted by a_1, a_2, a_3, and the particles at their ends are denoted by m_1, m_2, m_3. We shall define the position of the pendulum in the rotating frame of reference by the angles $\varphi_1,\varphi_2,\varphi_3$, measured anticlockwise from the downward vertical. We shall assume that $\varphi_1 \in [0, \pi]$, $\varphi_i \in (-\pi, \pi]$ $(i=2,3)$ (Fig. 1).

The changed potential of the mechanical system is made up of the gravitational force potential W_g and the potential W_e of the translational inertial forces:

$$W = W_g + W_e = -g\sum_{i=1}^{3} m_i \sum_{j=1}^{i} a_j \cos\varphi_j - \frac{\omega^2}{2}\sum_{i=1}^{3} m_i \left(\sum_{j=1}^{i} a_j \sin\varphi_j\right)^2$$

After introducing the dimensionless parameters

$$p = \frac{a_2}{a_1},\quad q = \frac{a_3}{a_1},\quad \mu = \frac{m_1+m_2+m_3}{m_3},\quad \nu = \frac{m_2+m_3}{m_3},\quad \Omega = \frac{\omega^2 a_1}{g}$$

$$\mu > \nu > 1,\quad \Omega > 0,\quad p > 0,\quad q > 0$$

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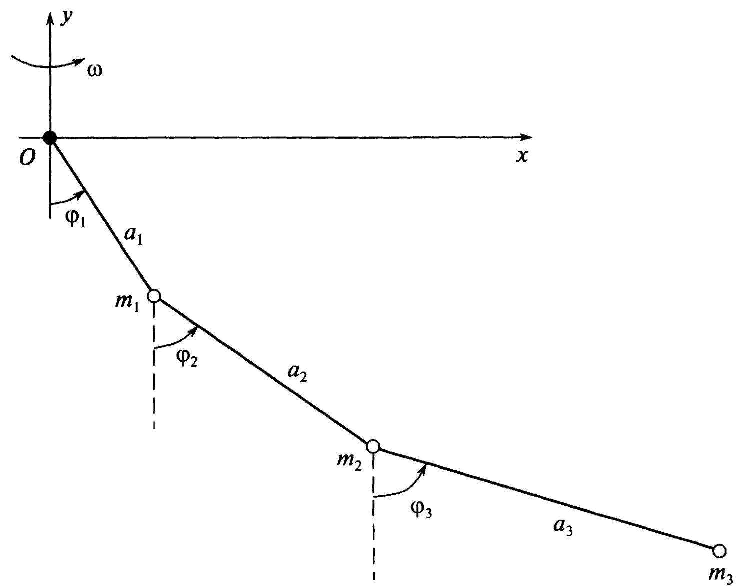

Fig. 1.

it can be represented in the form

$$V = \frac{W}{a_1 m_3 g} = -\mu\cos\varphi_1 - p\nu\cos\varphi_2 - q\cos\varphi_3 -$$
$$-\frac{1}{2}\Omega\left(\mu\sin^2\varphi_1 + p^2\nu\sin^2\varphi_2 + q^2\sin^2\varphi_3\right) -$$
$$-\Omega(p\nu\sin\varphi_1\sin\varphi_2 + pq\sin\varphi_2\sin\varphi_3 + q\sin\varphi_1\sin\varphi_3)$$

It is obvious that the number and character of the critical points of the functions W and V are identical, and we shall therefore subsequently study the function V.

2. Trivial equilibrium positions and their stability

The relative equilibrium positions of the mechanical system considered here correspond to the critical points of the function V,[14] which are determined from the system of equations

$$\frac{\partial V}{\partial\varphi_1} = 0, \quad \frac{\partial V}{\partial\varphi_2} = 0, \quad \frac{\partial V}{\partial\varphi_3} = 0 \qquad (2.1)$$

It is obvious that system (2.1) admits of trivial solutions of the form

$$\sin\varphi_i = 0 \quad i = 1,2,3$$

When account is taken of the intervals of variation of the angles φ_1, φ_2, φ_3 there are eight geometrically different trivial equilibrium positions which exist for any angular velocities. We shall denote these solutions by $S^{\pm\pm\pm}$, where the plus and minus sign is chosen if the cosine of the corresponding angle is equal to 1 (a rod is directed vertically downwards) or -1 (a rod is directed vertically upwards). Hence, writing S^{+-+} means that a solution is considered for which $\cos\varphi_1 = 1$, $\cos\varphi_2 = -1$, $\cos\varphi_3 = 1$, that is, the first and third rods are directed vertically downwards and the second rod is directed vertically upwards. The principal minor of order s ($s = 1, 2, 3$) of the matrix of the second derivatives of the function V, calculated for the trivial solution, is denoted by $\Delta_s(\Omega)$.

Considering the values of the minor Δ_3 for the roots of the minors $\Delta_{1,2}$ as well as for zero and at $+\infty$, it is possible, for each trivial solution, to determine the number of positive roots of the minor Δ_3 and their location relative to the roots of the minors $\Delta_{1,2}$ and to draw conclusions on the basis of this regarding the degree of instability of each of the trivial solutions.

Suppose Ω_1, Ω_2, Ω_3 are the roots of the minor $\Delta_3(\Omega)$. A brief description of the results of the investigations is presented below.

In the case of a solution S^{+++}, all three roots Ω_1, Ω_2, Ω_3 of the minor Δ_3 are positive, and the degree of instability of the solution S^{+++} is equal to 0 (the solution is stable) when $\Omega \in (0,\Omega_1)$ and equal to 1 when $\Omega \in (\Omega_1,\Omega_2)$, 2 when $\Omega \in (\Omega_2,\Omega_3)$ and 3 when $\Omega \in (\Omega_3,\infty)$.

The degree of instability of the solutions S^{++-}, S^{+-+}, S^{-++} is equal to 1 when $\Omega \in (0,\Omega_1)$, 2 when $\Omega \in (\Omega_1,\Omega_2)$ and 3 when $\Omega \in (\Omega_2,\infty)$. Of course, the roots Ω_1,Ω_2 have their own numerical values for each trivial solution S^{++-}, S^{+-+}, S^{-++}.

The degree of instability of the solutions S^{+--}, S^{-+-}, S^{--+} is equal to 2 when $\Omega \in (0,\Omega_1)$ and 3 when $\Omega \in (\Omega_1,\infty)$. As in the preceding case, the value of the root Ω_1 depends on the trivial solution considered.

The degree of instability of the solution S^{---} is equal to 3 for all $\Omega > 0$.

Hence, the degree of instability of the trivial solutions changes when the parameter Ω passes through the roots of the minor $\Delta_3(\Omega)$ and, consequently, according to bifurcation theory,[14] other non-trivial equilibrium positions branch off from these solutions: three non-trivial solutions branch off from the solution S^{+++}, two non-trivial solutions from each of the solutions S^{++-}, S^{+-+}, S^{-++} and one non-trivial solution from each of the solutions respectively.

3. Transformation of the equations

It is difficult to solve system (2.1) analytically, and it is therefore of interest to obtain asymptotic expansions of its solutions when $\Omega \to \infty$. In order to construct these expansions, it is convenient first to transform system (2.1). Introducing the parameter

$$\varepsilon = \frac{1}{\Omega} = \frac{g}{\omega^2 a_1}$$

we rewrite it in the matrix form

$$\begin{Vmatrix} \mu\varepsilon - \mu\cos\varphi_1 & -\nu p\cos\varphi_1 & -q\cos\varphi_1 \\ -\nu\cos\varphi_2 & \nu\varepsilon - \nu p\cos\varphi_2 & -q\cos\varphi_2 \\ -\cos\varphi_3 & -p\cos\varphi_3 & \varepsilon - q\cos\varphi_3 \end{Vmatrix} \begin{Vmatrix} \sin\varphi_1 \\ \sin\varphi_2 \\ \sin\varphi_3 \end{Vmatrix} = 0 \tag{3.1}$$

System (3.1) has a non-trivial solution $(\varphi_1(\varepsilon), \varphi_2(\varepsilon), \varphi_3(\varepsilon))$ for a fixed value of ε if and only if the rank of its matrix at the point $(\varepsilon, \varphi_1(\varepsilon), \varphi_2(\varepsilon), \varphi_3(\varepsilon))$ is less than three. Otherwise, all the values of $\sin\varphi_i$ are obliged to be zero values.[15]

We next show that the rank of its matrix is equal to two for all non-trivial solutions of system (3.1).

It can be verified by direct calculation that, in the case of system (3.1), a solution does not exist for which a certain row of the matrix consists of nothing but zeroes. It follows from this in particular that the rank of the matrix of system (3.1) cannot be equal to zero.

We next verify that a solution does not exist for which the rank of the matrix is equal to unity. We assume the opposite, that is, that the second row is proportional to the first row with a coefficient X and the third row is proportional to the first row with a coefficient Y. In this case, from the conditions of proportionality of the elements of the first column we obtain two equalities from which it follows that

$$X\nu\cos\varphi_2 = Y\cos\varphi_3$$

Similarly, from the second column of the matrix we have two equalities from which, when account is taken of relation obtained, it follows that $X=0$.

Hence, for any non-trivial solution, the rank of the matrix of system (3.1) is always equal to two, that is, the equalities

$$\begin{aligned} &-X\cos\varphi_3 - Y\nu\cos\varphi_2 = Z\mu(\varepsilon - \cos\varphi_1) \\ &-Xp\cos\varphi_3 + Y\nu(\varepsilon - p\cos\varphi_2) = -Z\nu p\cos\varphi_1 \\ &X(\varepsilon - q\cos\varphi_3) - Yq\cos\varphi_2 = -Zq\cos\varphi_1 \end{aligned} \tag{3.2}$$

hold for any non-trivial solution of system (3.1) in the case of a fixed ε.

We will now show that the coefficient Z never vanishes and it can be taken as unity.

In relations (3.2), we put $Z=0$. If $X=0$, it follows from equalities (3.2) that $Y=0$ also. This means that $X \neq 0$ and that it is possible to separate each of equations (3.2) in X. Subtracting the second equation, divided by p, from the first equation, we obtain

$$Y\nu\varepsilon/Xp = 0$$

It follows from this that $Y=0$ and we then obtain from relations (3.2) that $X=0$ also.

Consequently, $Z \neq 0$ and it can be assumed that $Z=1$.

Henceforth, in order to abbreviate the writing of the expansions of the solutions in the parameter ε, we shall use the notation

$$A_\nu = \nu - 1, \quad A_\mu = \mu - 1, \quad A_{\mu\nu} = \mu - \nu$$

In the same way as was done in the case of the coefficient Z, it can be shown that the coefficient X also cannot be equal to zero and that the coefficient Y vanishes only if the system parameters satisfy the equalities

$$q = \nu \quad \mu A_\nu / A_{\mu\nu} = 1$$

In this case, the solutions

$$\cos\varphi_1 = \frac{1}{\Omega A_\nu}, \quad \cos\varphi_2 = \pm 1, \quad \cos\varphi_3 = \frac{1}{\Omega A_\nu}, \quad \Omega > \frac{1}{A_\nu}$$

exist for system (3.1) for which $Y=0$. In the case of the other solutions of (3.1) and also in the case when the above equalities are not satisfied, the coefficient Y does not vanish for sufficiently small ε. It can therefore be assumed that $X \neq 0$, $Y \neq 0$ when $\Omega > \Omega_0$.

For the new variables X and Y we obtain the equations

$$X^2(Xp-Yq)^2 - X^2(p\mu+\nu Y)^2\frac{q^4A_\nu^2}{\nu^2A_{\mu\nu}^2}+$$
$$+\varepsilon^2\frac{q^2-X^2}{p^2A_{\mu\nu}^2}(p\mu+\nu Y)^2(Xp-Yq)^2=0$$
$$Y^2(p\mu+\nu Y)^2 - Y^2\left(Y\frac{\nu A_\mu}{pA_{\mu\nu}}-X\frac{1}{q}+\frac{\mu A_\nu}{A_{\mu\nu}}\right)^2\frac{A_{\mu\nu}^2}{A_\nu^2}-$$
$$-\varepsilon^2\frac{p^2-Y^2}{p^2A_\nu^2}(p\mu+\nu Y)^2\left(Y\frac{\nu A_\mu}{pA_{\mu\nu}}-X\frac{1}{q}+\frac{\mu A_\nu}{A_{\mu\nu}}\right)^2=0 \tag{3.3}$$

Equations (3.3) determine the values of X and Y after which it is possible to recover the values of $\cos\phi_i$ from the relations

$$\cos\varphi_1=\frac{p\mu+\nu Y}{pA_{\mu\nu}}\varepsilon,\quad \cos\varphi_2=\frac{Yq\nu A_\mu - XpA_{\mu\nu}+pq\mu A_\nu}{YpqA_\nu A_{\mu\nu}}\varepsilon$$
$$\cos\varphi_3=\frac{\nu(Xp-Yq)}{XpqA_\nu}\varepsilon \tag{3.4}$$

and, in order to determine the values of the angles themselves, it is necessary, moreover, to establish the signs of $\sin\phi_i$. Taking account of the fact that $\sin\phi_1>0$, we obtain

$$\operatorname{sgn}(\sin\varphi_2)=\operatorname{sgn}(-Y)\operatorname{sgn}(\cos\varphi_1)\operatorname{sgn}(\cos\varphi_2)$$
$$\operatorname{sgn}(\sin\varphi_3)=\operatorname{sgn}(-X)\operatorname{sgn}(\cos\varphi_1)\operatorname{sgn}(\cos\varphi_3) \tag{3.5}$$

4. Limiting solutions of system (3.3)

Putting $\varepsilon=0$ in Eqs (3.3), we obtain the following limiting values when $\varepsilon\to 0$ of the variables X and Y (the characteristic limit points)

$$P_0=(0,0),\quad P_{1,2}=\left(0,-\frac{p\mu(p\pm1)A_\nu}{\nu(pA_\nu\pm A_\mu)}\right),\quad P_{3,4}=\left(\pm\frac{q^2\mu A_\nu}{\nu A_{\mu\nu}},0\right)$$
$$P_{5,6,7,8}=\left(-\frac{q\mu(1-\kappa_1q-\kappa_2p)}{(\mu-\kappa_1q-\kappa_2p\nu)},-\frac{p\mu(\nu-\kappa_1q-\kappa_2p\nu)}{\nu(\mu-\kappa_1q-\kappa_2p\nu)}\right),\quad \kappa_{1,2}=\pm1$$

The upper (lower) signs correspond to the points P_1 and P_3 (P_2 and P_4).

To be specific, we shall assume that the following values of κ_1 and κ_2

P_5	P_6	P_7	P_8
$\kappa_1=1,\kappa_2=1$	$\kappa_1=1,\kappa_2=-1$	$\kappa_1=-1,\kappa_2=1$	$\kappa_1=-1,\kappa_2=-1$

correspond to the points $P_5,\ldots,P_8$.

Limit points can be identical or not exist for some parameter values. For instance, when $p=1$ the point P_2 coincides with the point P_0 and it does not exist when $p=A_\mu/A_\nu$.

In the case of the points $P_5,\ldots,P_8$, if the numerator of any of the fractions vanishes this means that the corresponding point coincides with one of the points $P_0,\ldots,P_4$.

If, however, the denominator vanishes for any p, q, κ_1, κ_2 in the expressions for $P_5,\ldots,P_8$, then the corresponding point does not exist.

It should be noted that, apart from the limiting values of the variables X and Y found, solutions of system (3.3) exist for which one or both of the variables tends to infinity when $\varepsilon\to 0$.

5. Expansions of the solutions of system (3.3) branching off from the characteristic limit points and their stability

5.1. Asymptotic solutions branching off from the points $P_5,\ldots,P_8$

Expanding the solutions of system (3.3) in series in natural powers of ε, it is possible to show that one solution of system (6.3) of the form

$$X=X_0+O(\varepsilon),\quad Y=Y_0+O(\varepsilon)$$

branches off from each point $P_5,\ldots,P_8$, where X_0 and Y_0 are the first and second coordinates of the point $P_i (i=5,\ldots,8)$ respectively.

For all the solutions considered, the links of the pendulum tend to take up a horizontal position as the angular velocity increases. However, the mutual arrangement of the links and their degree of instability depend on a specific limit point, that is, on the signs of κ_1 and κ_2. The subdivisions of the plane of the parameters (p,q) into domains, where the degree of instability does not change, are represented in Figs. 2-4. The approximate shape of the corresponding configuration of the pendulum in each domain is also shown.

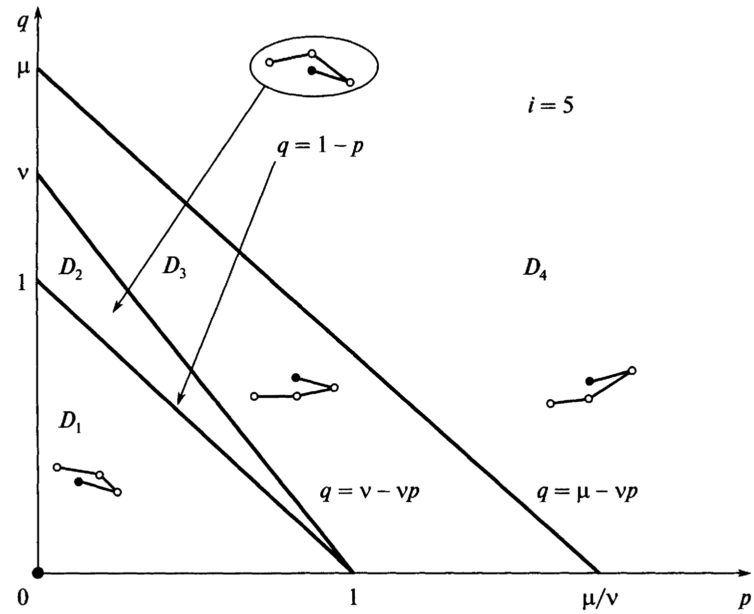

Fig. 2.

The solution $(X(\varepsilon), Y(\varepsilon))$ branching off from the point P_8 is stable and the pendulum tends to a position when all the links are stretched out in tandem along a horizontal axis.

In the case of the point P_5, the plane of the parameters (p, q) is subdivided into four domains (Fig. 2) in which the solution branching off from this point has different degrees of instability: 2, if $(p,q) \in D_1$, 1, if $(p,q) \in D_2$ or $(p,q) \in D_4$ and, when $(p,q) \in D_3$, the solution is stable. The approximate arrangement of the pendulum links for this solution in each domain is also shown in Fig. 2.

For the point $P_{6,}$ the domains with a constant degree of instability, into which the plane of the parameters (p,q) is divided, are shown in Fig. 3. The degree of instability of the solution is equal to 2 , if $(p,q) \in D_2$, 1, if $(p,q) \in D_1$ or $(p,q) \in D_3$, and the solution is stable when $(p,q) \in D_2$.

In the case of the point P_7, the plane of the parameters (p, q) is divided into five domains (Fig. 4). The degree of instability of the solution equals 2, if $(p,q) \in D_1$, 1, if $(p,q) \in D_2, D_3, D_5$, and the solution is stable if $(p,q) \in D_4$.

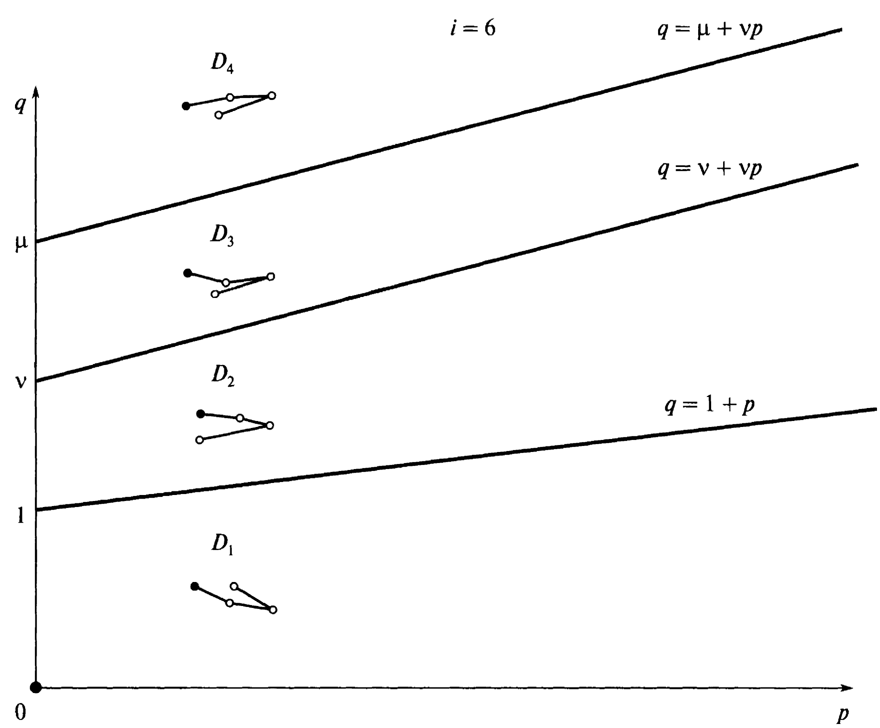

Fig. 3.

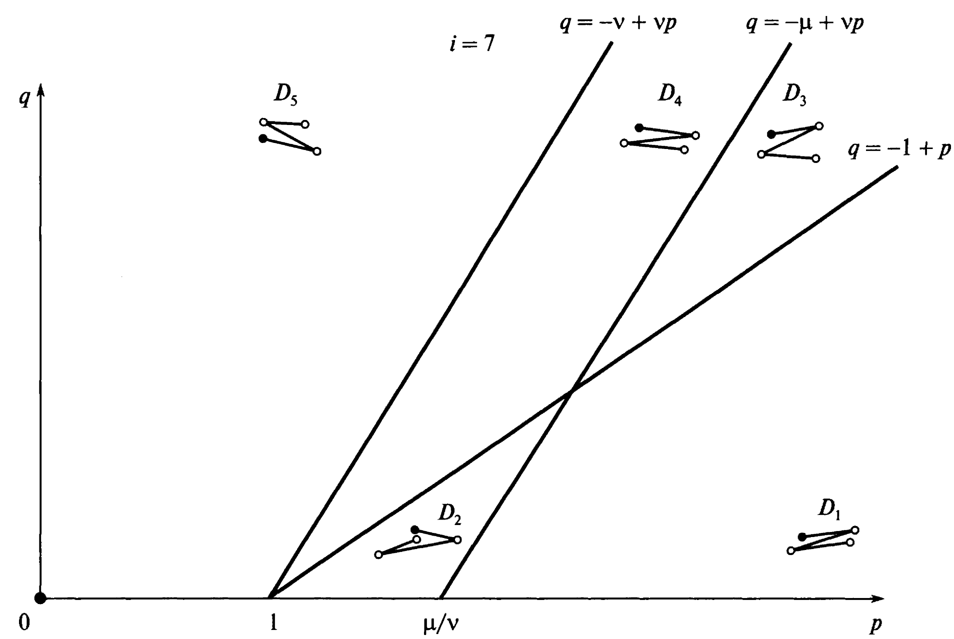

Fig. 4.

5.2. Asymptotic solutions of system (3.3) branching off from the point P_0

We will describe the method of investigating the branching off of solutions from the limit points which have been found , taking the point P_0 as an example. Suppose

$$X = B\varepsilon^a, \quad Y = A\varepsilon^b$$

In Eqs. (3.3), we write out the terms tending to zero at the slowest rate

$$-X^2\frac{q^4\mu^2p^2A_\nu^2}{\nu^2A_{\mu\nu}^2} - 2\varepsilon^2XY\frac{q^3p\mu^2}{A_{\mu\nu}^2} + \varepsilon^2Y^2\frac{q^4\mu^2}{A_{\mu\nu}^2} + \ldots = 0$$

$$-\varepsilon^2\frac{\mu^4p^2}{A_{\mu\nu}^2} + Y^2\mu^2(p^2-1) + \ldots = 0 \tag{5.1}$$

When $p > 1$, it follows from the second equation that $b = 1$ and the coefficient A is determined from the equation

$$-\frac{\mu^4p^2}{A_{\mu\nu}^2} + A^2\mu^2(p^2-1) = 0$$

which has two solutions

$$A = \pm\frac{p\mu}{A_{\mu\nu}(p^2-1)^{1/2}}$$

When $p \le 1$, this equation does not have any solutions.

If, in the first equation of (5.1), $X \sim \varepsilon^2 XY$, then $a = 3$ and the term $\varepsilon^2 Y^2$, which is of the order of ε^4, is not cancelled out by any other term. This means that $X \sim \varepsilon^2 Y^2$ and $a = 2$ and the coefficient B is determined from the equation

$$-B^2\frac{q^4\mu^2p^2A_\nu^2}{\nu^2A_{\mu\nu}^2} + \frac{p^2\mu^2}{A_{\mu\nu}^2(p^2-1)}\frac{q^4\mu^2}{A_{\mu\nu}^2} = 0$$

There are four solutions branching off from the point (0,0):

$$X = \kappa_1\frac{\mu\nu}{A_\nu A_{\mu\nu}}(p^2-1)^{-1/2}\varepsilon^2 + O(\varepsilon^3)$$

$$Y = \kappa_2\frac{p\mu}{A_{\mu\nu}}(p^2-1)^{-1/2}\varepsilon + O(\varepsilon^2); \quad \kappa_{1,2} = \pm 1$$

Suppose $p = 1$ (in which case the coefficient of Y^2 vanishes). Equations (3.3) take the form

$$-X^2\frac{q^4\mu^2A_\nu^2}{\nu^2A_{\mu\nu}^2}-2\varepsilon^2XY\frac{q^3\mu^2}{A_{\mu\nu}^2}+\varepsilon^2Y^2\frac{q^4\mu^2}{A_{\mu\nu}^2}+\ldots=0$$
$$-\varepsilon^2\frac{\mu^4}{A_{\mu\nu}^2}-2Y^3\mu\nu\frac{A_{\mu\nu}}{A_\nu}+2XY^2\frac{\mu A_{\mu\nu}}{qA_\nu}+\ldots=0 \tag{5.2}$$

The indices a and b are found from the conditions that the powers for ε are equal for any two of the three terms X^2, ε^2XY, ε^2Y^2 in the first equation of (5.2) and XY^2, Y^3, ε^2 in the second equation, with the additional condition that the power of the remaining terms will not be lower.

A treatment of all possible cases leads to the unique solution

$$a=5/3,\quad b=2/3$$

To determine of A and B, we have the system of equations

$$-A^2\frac{q^4\mu^2A_\nu^2}{\nu^2A_{\mu\nu}^2}+B^2\frac{q^4\mu^2}{A_{\mu\nu}^2}=0,\quad -\frac{\mu^4}{A_{\mu\nu}^2}-2B^3\mu\nu\frac{A_{\mu\nu}}{A_\nu}=0$$

As a result, we obtain the two solutions

$$X=-\kappa\frac{\mu\nu}{A_\nu A_{\mu\nu}}\left(\frac{A_\nu}{2\nu}\right)^{1/3}\varepsilon^{5/3}+O(\varepsilon^{7/3}),\quad \kappa=\pm1$$
$$Y=-\frac{\mu}{A_{\mu\nu}}\left(\frac{A_\nu}{2\nu}\right)^{1/3}\varepsilon^{2/3}+O(\varepsilon^{4/3})$$

The mutual arrangement of the links of the pendulum, corresponding to the solutions found, will henceforth be described in the following form.

The position of the first rod is completely defined by the expression for $\cos\varphi_1$ while, for a single-valued determination of the position of the second and third rods, in addition to the expressions for the cosines of the angles it is necessary to know the signs of the sines.

Then, using relations (3.4), it is possible to obtain asymptotic formulae for the cosines of the angles of deflection of the pendulum: the signs of the sines of these angles are determined from relations (3.5).

In the case of the limit point P_0 considered, we obtain

$$\cos\varphi_1=\varepsilon\frac{\mu}{A_{\mu\nu}}+O(\varepsilon^2)$$

In the case when $p\neq1$

$$\cos\varphi_2=\kappa_2\left(1-\frac{1}{p^2}\right)^{1/2}+O(\varepsilon),\quad \sin\varphi_2=-\frac{1}{p}+O(\varepsilon)$$
$$\cos\varphi_3=-\kappa_1\kappa_2+O(\varepsilon),\quad \sin\varphi_3=O(\varepsilon)>0$$

In the case when $p = 1$

$$\cos\varphi_2=-\left(\frac{2\nu}{A_\nu}\right)^{1/3}\varepsilon^{1/3}+O(\varepsilon^{2/3}),\quad \sin\varphi_2=-1+O(\varepsilon^{1/3})$$
$$\cos\varphi_3=-\kappa+O(\varepsilon^{1/3}),\quad \sin\varphi_3=O(\varepsilon^{1/3})<0$$

The stability of all the solutions found is analysed by calculating the second derivatives of the function V in the principal approximation with respect to ε.

The solution obtained above have a degree of instability of 2 in both cases.

All the remaining limit points $P_1,\ldots,P_4$, and also the solutions for which at least one of the variables is unbounded when $\varepsilon\to0$, are investigated using a similar method. The results of the analysis performed are presented below.

5.3. Asymptotic solutions branching off from the point P_1

In order to shorten the form of the asymptotic expansions, we will henceforth denote the first coordinate of a point P_i, by X_0 and the second coordinate by Y_0.

We introduce the notation

$$r_\pm=(p\pm1)/q$$

As was done in the preceding Subsection, it can be shown that solutions, branching off from the point P_1, exist if the inequality $r_+ \leq 1$ is satisfied.

In the case when this inequality is rigorous, two solutions branch out from the point P_1

$$X = \kappa \frac{(p+1)\mu}{(pA_\nu + A_\mu)}\left(1 - r_+^2\right)^{-1/2}\varepsilon + O(\varepsilon^2)$$

$$Y = Y_0 + \kappa \frac{p\mu A_{\mu\nu} r_+}{\nu(pA_\nu + A_\mu)^2}\left(1 - r_+^2\right)^{-1/2}\varepsilon + O(\varepsilon^2); \quad \kappa = \pm 1$$

From relations (3.4), we obtain their geometric interpretation

$$\cos\varphi_1 = \varepsilon \frac{\mu}{pA_\nu + A_\mu} + O(\varepsilon^2)$$

$$\cos\varphi_2 = \varepsilon \frac{\nu}{(p+1)A_\nu} + O(\varepsilon^2), \quad \sin\varphi_2 = 1 + O(\varepsilon)$$

$$\cos\varphi_3 = \kappa\left(1 - r_+^2\right)^{1/2} + O(\varepsilon), \quad \sin\varphi_3 = -r_+ + O(\varepsilon)$$

If, however, the equality $r_+ = 1$,[1] is satisfied, then one solution

$$X = \frac{(p+1)\mu}{(pA_\nu + A_\mu)}\left(\frac{p+1}{2}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon^{4/3})$$

$$Y = Y_0 + \frac{p\mu A_{\mu\nu}}{\nu(pA_\nu + A_\mu)^2}\left(\frac{p+1}{2}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon^{4/3})$$

branches off from the limit point P_1.

Its geometric interpretation is

$$\cos\varphi_1 = \varepsilon \frac{\mu}{pA_\nu + A_\mu} + O(\varepsilon^{4/3})$$

$$\cos\varphi_2 = \varepsilon \frac{\nu}{(p+1)A_\nu} + O(\varepsilon^{5/3}), \quad \sin\varphi_2 = 1 + O(\varepsilon)$$

$$\cos\varphi_3 = \left(\frac{p+1}{2}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon), \quad \sin\varphi_3 = -1 + O(\varepsilon^{1/3})$$

In both cases, the degree of instability of the solutions is equal to 1.

5.4. *Asymptotic solutions branching off from the point P_2. Two solutions*

$$X = \kappa \frac{|p-1|\mu}{|pA_\nu - A_\mu|}\left(1 - r_-^2\right)^{-1/2}\varepsilon + O(\varepsilon^2)$$

$$Y = Y_0 - \kappa \frac{|r_-|p\mu A_{\mu\nu}}{\nu|pA_\nu - A_\mu|(pA_\nu - A_\mu)}\left(1 - r_-^2\right)^{-1/2}\varepsilon + O(\varepsilon^2); \quad \kappa = \pm 1$$

branch off from the limit point P_2 in the case when $|r_-| < 1$.

The configuration of the pendulum for large angular velocities is defined by the relations

$$\cos\varphi_1 = -\varepsilon \frac{\mu}{pA_\nu - A_\mu} + O(\varepsilon)$$

$$\cos\varphi_2 = \varepsilon \frac{\nu}{(p-1)A_\nu} + O(\varepsilon^2), \quad \sin\varphi_2 = -1 + O(\varepsilon)$$

$$\cos\varphi_3 = \kappa\,\mathrm{sgn}(r_-(pA_\nu - A_\mu))\left(1 - r_-^2\right)^{1/2} + O(\varepsilon), \quad \sin\varphi_3 = r_- + O(\varepsilon)$$

The degree of instability of the solutions depends on the value of p: it is equal to 2, if $p \in V_{(0,1)} \cup (A_\mu/A_\nu, +\infty)$, and 1, if $p \in (1, A_\mu/A_\nu)$.

When the equality $r_- = 1$ is satisfied, the solution

$$X = -\frac{(p-1)\mu}{(pA_\nu - A_\mu)}\left(\frac{(p-1)^2}{2|p-1|}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon)$$

$$Y = Y_0 + \frac{p(p-1)\mu A_{\mu\nu}}{|p-1|\nu(pA_\nu - A_\mu)^2}\left(\frac{(p-1)^2}{2|p-1|}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon)$$

$$\cos\varphi_1 = -\varepsilon\frac{\mu}{pA_\nu - A_\mu} + O(\varepsilon^{4/3})$$

$$\cos\varphi_2 = \varepsilon\frac{\nu}{(p-1)A_\nu} + O(\varepsilon^{4/3}), \quad \sin\varphi_2 = -1 + O(\varepsilon^{1/3})$$

$$\cos\varphi_3 = -\left(\frac{2|p-1|}{(p-1)^2}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon), \quad \sin\varphi_3 = \operatorname{sgn}(p-1) + O(\varepsilon^{1/3})$$

branches off from the limit point.

The distribution of the degrees of instability as a function of the value of the parameter p remains the same as in the preceding case.

If, however, the inequality $r_- > 1$ is satisfied, then the solutions branching off from this limit point do not exist.

5.5. Asymptotic solutions branching off from the point P_3

We introduce the notation

$$r_{\pm\nu} = (q \pm \nu)/\nu p$$

Solutions, branching off from the point P_3, exist when $r_{+\nu} \le 1$.

If this inequality is rigorous, then there are two solutions:

when $q \ne A_{\mu\nu}/A_\nu$

$$X = X_0 + \kappa\frac{(qA_\nu - A_{\mu\nu})r_{+\nu}}{A_{\mu\nu}^2}\left(1 - r_{+\nu}^2\right)^{-1/2}\varepsilon + O(\varepsilon^2)$$

$$Y = \kappa\frac{p\mu}{A_{\mu\nu}}r_{+\nu}\left(1 - r_{+\nu}^2\right)^{-1/2}\varepsilon + O(\varepsilon^2); \quad \kappa = \pm 1$$

when $q = A_{\mu\nu}/A_\nu$

$$X = X_0 + \frac{\mu}{2\nu A_\nu}\left(1 - \frac{\mu^2}{\nu^2}\right)\varepsilon^2 + O(\varepsilon^3)$$

$$Y = \kappa\frac{p\mu}{A_{\mu\nu}}r_{+\nu}\left(1 - r_{+\nu}^2\right)^{-1/2}\varepsilon + O(\varepsilon^3), \quad \kappa = \pm 1$$

The configuration of the pendulum is defined by the expressions

$$\cos\varphi_1 = \frac{\mu}{A_{\mu\nu}}\varepsilon + O(\varepsilon^2)$$

$$\cos\varphi_2 = \kappa\left(1 - r_{+\nu}^2\right)^{1/2} + O(\varepsilon), \quad \sin\varphi_2 = -r_{+\nu} + O(\varepsilon)$$

$$\cos\varphi_3 = \frac{\nu}{qA_\nu}\varepsilon + O(\varepsilon^2), \quad \sin\varphi_3 = 1 + O(\varepsilon)$$

If, however, the equality $r_{+\nu} = 1$ is satisfied, then a unique solution of system (3.3) branches off from the limit point P_3:

when $q \neq A_{\mu\nu}/A_\nu$

$$X = X_0 + \frac{\mu\nu(p-1)}{A_{\mu\nu}^2}(p\nu A_\nu - \nu A_\nu - A_{\mu\nu})\left(\frac{p}{2}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon)$$

$$Y = -\frac{p\mu}{A_{\mu\nu}}\left(\frac{p}{2}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon)$$

when $q = A_{\mu\nu}/A_\nu$

$$X = X_0 + \frac{\mu}{2\nu A_\nu}\left(1 - \frac{\mu^2}{\nu^2}\right)\varepsilon^2 + O(\varepsilon^{7/3})$$

$$Y = -\frac{\mu(\nu A_\nu + A_{\mu\nu})}{\nu A_\nu A_{\mu\nu}}\left(\frac{\nu A_\nu + A_{\mu\nu}}{2\nu A_\nu}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon)$$

In this case, we obtain from relations (3.4)

$$\cos\varphi_1 = \frac{\mu}{A_{\mu\nu}}\varepsilon + O(\varepsilon^{4/3})$$

$$\cos\varphi_2 = -\left(\frac{p}{2}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon^{2/3}), \quad \sin\varphi_2 = -1 + O(\varepsilon^{1/3})$$

$$\cos\varphi_3 = \frac{\nu}{qA_\nu}\varepsilon + O(\varepsilon^{4/3}), \quad \sin\varphi_3 = 1 + O(\varepsilon)$$

The degree of instability of the solutions is equal to 1 in all cases.

5.6. Remark

Although the form of the asymptotics for the variables X and Y depends on the ratio of q to $A_{\mu\nu}/A_\nu$, neither the stability nor the geometrical illustration of the solutions depend on this ratio (it does affect the rate at which the links of the pendulum tend to the limiting configuration).

5.7. Asymptotic solutions branching off from the point P_4

System (3.3) has solutions branching off from this point when the inequality $|r_{-\nu}| \leq 1$ is satisfied.

If this is a strict inequality, then two solutions branch off from the point P_4:

when $q \neq \nu$

$$X = X_0 + \kappa\frac{(qA_\nu + A_{\mu\nu})|r_{-\nu}|}{\nu A_{\mu\nu}^2}\left(1 - r_{-\nu}^2\right)^{-1/2}\varepsilon + O(\varepsilon^2)$$

$$Y = \kappa\frac{p\mu}{A_{\mu\nu}}|r_{-\nu}|\left(1 - r_{-\nu}^2\right)^{-1/2}\varepsilon + O(\varepsilon^2), \quad \kappa = \pm 1$$

$$\cos\varphi_1 = \frac{\mu}{A_{\mu\nu}}\varepsilon + O(\varepsilon^2)$$

$$\cos\varphi_2 = -\kappa\,\mathrm{sgn}\,r_{-\nu}\left(1 - r_{-\nu}^2\right)^{1/2} + O(\varepsilon), \quad \sin\varphi_2 = r_{-\nu} + O(\varepsilon)$$

$$\cos\varphi_3 = \frac{\nu}{qA_\nu}\varepsilon + O(\varepsilon^2), \quad \sin\varphi_3 = -1 + O(\varepsilon)$$

when $q = \nu$

$$X = X_0 + \frac{\mu\nu}{2A_\nu A_{\mu\nu}}\left|1 - \frac{\mu^2 A_\nu^2}{A_{\mu\nu}^2}\right|\varepsilon^2 + O(\varepsilon^3)$$
$$Y = \kappa\frac{\mu}{2A_\nu^2 A_{\mu\nu}}\left|1 - \frac{\mu^2 A_\nu^2}{A_{\mu\nu}^2}\right|\varepsilon^3 + O(\varepsilon^4), \quad \kappa = \pm 1$$
$$\cos\varphi_1 = \frac{\mu}{A_{\mu\nu}}\varepsilon + O(\varepsilon^2)$$
$$\cos\varphi_2 = -\kappa\,\mathrm{sgn}\left(\frac{\mu^2 A_\nu^2}{A_{\mu\nu}^2} - 1\right) + O(\varepsilon)$$
$$\cos\varphi_3 = \frac{\varepsilon}{A_\nu} + O(\varepsilon^2), \quad \sin\varphi_3 = -1 + O(\varepsilon)$$

where the sign of $\sin\varphi_2$ depends on the relation between $\mu A_\nu/A_{\mu\nu}$ and 1. When

$$\mu A_\nu/A_{\mu\nu} < 1$$

$\sin\varphi_2 = O(\varepsilon) < 0$, and satisfaction of the opposite inequality implies that $\sin\varphi_2 = O(\varepsilon) > 0$.

The degree of instability of the solutions is equal to 1 in all cases.

Next, suppose $|r_{-\nu}| = 1$.

The, the unique solution

$$X = X_0 + \frac{p(1+\kappa p)\mu\nu}{A_{\mu\nu}^2}\left(\frac{A_{\mu\nu}}{p} + \nu\left(1 + \kappa\frac{A_\mu}{pA_\nu}\right)\right)\left(\frac{p}{2}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon)$$
$$Y = -\frac{p\mu}{A_{\mu\nu}}\left(\frac{p}{2}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon); \kappa = \mathrm{sgn}\, r_{-\nu}$$

$$\cos\varphi_1 = \frac{\mu}{A_{\mu\nu}}\varepsilon + O(\varepsilon^{4/3})$$
$$\cos\varphi_2 = \mathrm{sgn}\, r_{-\nu}\left(\frac{p}{2}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon^{2/3}), \quad \sin\varphi_2 = \mathrm{sgn}\, r_{-\nu} + O(\varepsilon^{1/3})$$
$$\cos\varphi_3 = \frac{\nu}{qA_\nu}\varepsilon + O(\varepsilon^{4/3}), \quad \sin\varphi_3 = -1 + O(\varepsilon)$$

branches out from the point P_4.

The degree of instability of the solutions is equal to 1.

6. Asymptotic solution for which $X \to 0$, $Y \to \infty$ when $\varepsilon \to 0$

We will now show that solutions do not exist for which $X \to \infty$, $Y \to Y_0$ when $\varepsilon \to 0$. Actually, by assuming the opposite, we obtain that, in the first equation of system (3.3), the term X^4, which is not cancelled out by any other term, has the greatest order of growth.

It can then be shown using the method described in Section 5 that solutions, for which the variable X tends to zero and the variable Y tends to infinity when $\varepsilon \to 0$, exist if the following relations are simultaneously satisfied

$$p = A_\mu/A_\nu, \quad q > A_{\mu\nu}/A_\nu$$

In this case, there are two solutions for system (3.3)

$$X = -\kappa q(q^2A_\nu^2 - A_{\mu\nu}^2)^{-1/2}\frac{A_{\mu\nu}}{A_\mu}\left(2\mu A_\mu^2\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon^{2/3}), \quad \kappa = \pm 1$$

$$Y = -\frac{A_{\mu\nu}}{\nu A_\nu}\left(2\mu A_\mu^2\right)^{1/3}\varepsilon^{-2/3} + O(\varepsilon^{-1/3})$$

$$\cos\varphi_1 = -\left(\frac{2\mu}{A_\mu}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon^{2/3})$$

$$\cos\varphi_2 = \frac{\nu}{A_{\mu\nu}}\varepsilon + O(\varepsilon^{4/3}), \quad \sin\varphi_2 = -1 + O(\varepsilon)$$

$$\cos\varphi_3 = -\kappa\left(1 - \frac{A_{\mu\nu}^2}{q^2A_\nu^2}\right)^{1/2} + O(\varepsilon), \quad \sin\varphi_3 = \frac{A_{\mu\nu}}{qA_\nu} + O(\varepsilon)$$

All solutions have a degree of stability equal to 2.

7. Asymptotic solutions for which $X \to A \neq 0$, $Y \to \infty$ when $\varepsilon \to 0$

If the parameters of system (3.3) simultaneously satisfy the two inequalities

$$p < A_\mu/A_\nu, \quad q > pA_{\mu\nu}/A_\mu$$

it has four solutions

$$X = \kappa_1\frac{qA_{\mu\nu}}{A_\nu}\left(\frac{p^2A_\nu^2 - A_\mu^2}{p^2A_{\mu\nu}^2 - q^2A_\mu^2}\right)^{1/2} + O(\varepsilon)$$

$$Y = \kappa_2\frac{pA_{\mu\nu}}{\nu A_\mu}\left(A_\mu^2 - p^2A_\nu^2\right)^{1/2}\varepsilon^{-1} + O(1); \quad \kappa_{1,2} = \pm 1$$

$$\cos\varphi_1 = \kappa_2\left(1 - \frac{p^2A_\nu^2}{A_\mu^2}\right)^{1/2} + O(\varepsilon), \quad \sin\varphi_1 = \frac{pA_\nu}{A_\mu}$$

$$\cos\varphi_2 = \frac{\nu A_\mu}{pA_\nu A_{\mu\nu}}\varepsilon + O(\varepsilon^2), \quad \sin\varphi_2 = -1 + O(\varepsilon)$$

$$\cos\varphi_3 = -\kappa_1\kappa_2\left(1 - \frac{p^2A_{\mu\nu}^2}{q^2A_\mu^2}\right)^{1/2} + O(\varepsilon), \quad \sin\varphi_3 = \frac{pA_{\mu\nu}}{qA_\mu} + O(\varepsilon)$$

The degree of instability of all the solutions is equal to 2.
If, however,

$$p = A_\mu/A_\nu, \quad q = A_{\mu\nu}/A_\nu$$

then system (3.3) has a unique solution

$$X = \frac{A_{\mu\nu}}{A_\nu}T_0 + O(\varepsilon^{2/3})$$

$$Y = -\frac{A_{\mu\nu}}{\nu A_\nu}\left(2A_\mu^2(T_0 + \mu)\right)^{1/3}\varepsilon^{-2/3} + O(1)$$

$$\cos\varphi_1 = -\left(\frac{2(T_0 + \mu)}{A_\mu}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon^{2/3})$$

$$\cos\varphi_2 = \frac{\nu}{A_{\mu\nu}}\varepsilon + O(\varepsilon^{4/3}), \quad \sin\varphi_2 = -1 + O(\varepsilon)$$

$$\cos\varphi_3 = \left(\frac{2(T_0 + \mu)}{A_\mu}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon^{2/3}), \quad \sin\varphi_3 = 1 + O(\varepsilon^{1/3})$$

where T_0 is the unique real root of the equation

$$T^3 + T^2 + T\frac{A_{\mu\nu}}{\mu A_\nu} + \frac{A_{\mu\nu}}{A_\nu} = 0$$

The degree of instability of all the solutions is equal to 2.

When the values of the parameters of system (3.3) differ from those considered above, solutions of the above mentioned type do not exist.

8. Asymptotic solutions for which both variables are unbounded when $\varepsilon \to 0$

We introduce the notation

$$s_1 = \frac{q + \nu p}{\mu}, \quad s_2 = \frac{q - \nu p}{\mu}$$

and consider the case when

$$X \to \infty, \quad Y \to \infty, \quad X/Y \to A_0 \neq 0 \text{ when } \varepsilon \to 0$$

that is, $|X|$ and $|Y|$ have the same order of growth when $\varepsilon \to 0$. Such solutions exist when one or both of the following inequalities are satisfied

$$|s_i| < 1 \quad i = 1,2 \tag{8.1}$$

For each $i = 1, 2$ for which the corresponding value of s_i satisfies inequalities (8.1), system (3.3) has two solutions

$$X = \kappa_2 \frac{q(qA_\mu - (-1)^i pA_{\mu\nu})}{\mu s_i}\left(1 - s_i^2\right)^{1/2} \varepsilon^{-1} + O(1)$$

$$Y = \kappa_2 \frac{pA_{\mu\nu}}{\nu}\left(1 - s_i^2\right)^{1/2} \varepsilon^{-1} + O(1), \quad \kappa_2 = \pm 1$$

$$\cos\varphi_1 = \kappa_2\left(1 - s_i^2\right)^{1/2} + O(\varepsilon)$$

$$\cos\varphi_2 = (-1)^i \frac{\nu}{A_{\mu\nu} s_i}\varepsilon + O(\varepsilon^2)$$

$$\cos\varphi_3 = \frac{\mu}{qA_\mu - (-1)^i pA_{\mu\nu}}\varepsilon + O(\varepsilon^2)$$

When the inequality for s_2 is satisfied, the conditions

$$q \neq p\nu, \quad q \neq pA_{\mu\nu}/A_\mu$$

are the additional conditions for solutions corresponding to it to exist.

As a result of the investigation of the signs of the sines of the angles φ_1, φ_2, φ_3 and the stability of the solutions found, the plane of the parameters (p, q) is subdivided into three domains (Fig. 5).

For $i = 1$

$$\sin\varphi_2 = -1 + O(\varepsilon), \quad \sin\varphi_3 = -1 + O(\varepsilon)$$

The degree of instability of both solutions is equal to 1.

For $i = 2$

$$\sin\varphi_2 = -1 + O(\varepsilon), \quad \sin\varphi_3 = 1 + O(\varepsilon) \quad \text{when} \quad (p,q) \in D_1 \quad \text{or} \quad (p,q) \in D_2$$
$$\sin\varphi_2 = 1 + O(\varepsilon), \quad \sin\varphi_3 = -1 + O(\varepsilon) \quad \text{when} \quad (p,q) \in D_3$$

The degree of instability of the solutions is equal to 2, if $(p,q) \in D_1$ or $(p,q) \in D_3$, and it is equal to 1 if $(p,q) \in D_2$.

We will now assume that $|s_i| = 1$ for $i = 1$ or $i = 2$, that is, one of the inequalities (8.1) becomes an equality. In this case, instead of the corresponding two solutions for system (3.3), just one exists for which when $\varepsilon \to 0$

$$\varepsilon X \to 0, \quad \varepsilon Y \to 0, \quad \frac{X}{Y} \to A_0 \neq 0$$

$$X = -A_0 \frac{pA_{\mu\nu}}{\nu}\sqrt[3]{2}\varepsilon^{-2/3} + O(\varepsilon^{-1/3}), \quad Y = \frac{pA_{\mu\nu}}{\nu}\sqrt[3]{2}\varepsilon^{-2/3} + O(\varepsilon^{-1/3})$$

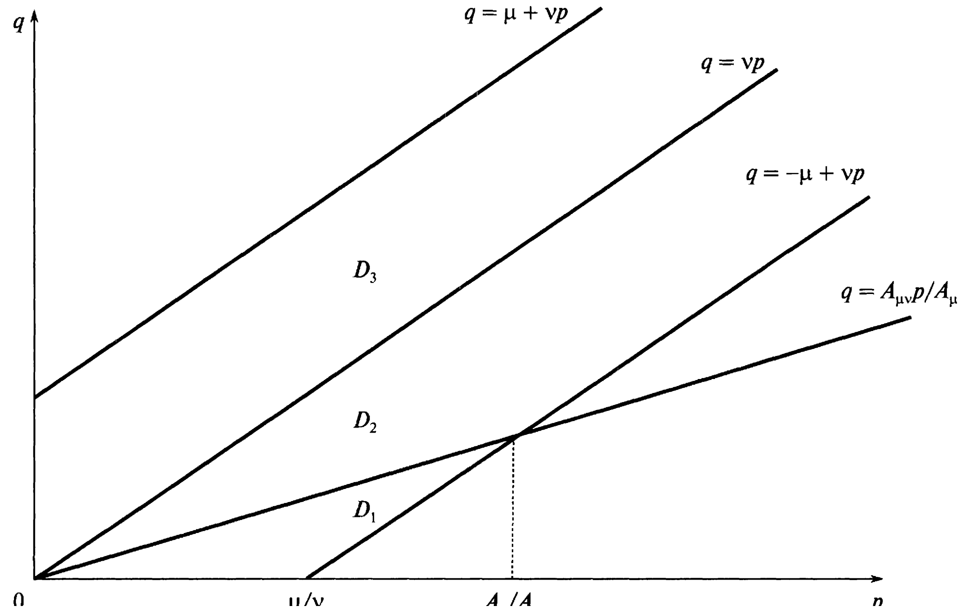

Fig. 5.

where

$$A_0 = \frac{q}{p} + (-1)^i \frac{q^2 A_\nu}{p A_{\mu\nu}}$$

$$\cos\varphi_1 = -\sqrt[3]{2}\varepsilon^{1/3} + O(\varepsilon^{2/3})$$

$$\cos\varphi_2 = -\operatorname{sgn} s_i \frac{\nu}{A_{\mu\nu}}\varepsilon + O(\varepsilon^{4/3})$$

$$\cos\varphi_3 = (-1)^i \frac{\nu}{A_{\mu\nu} + (-1)^i q A_\nu}\varepsilon + O(\varepsilon^{4/3})$$

The signs of the sines of the angles of deflection are as follows for these solutions:

$$\sin\varphi_2 = 1 + O(\varepsilon), \quad \sin\varphi_3 = -1 + O(\varepsilon), \quad \text{if} \quad s_1 = 1$$

$$\sin\varphi_2 = -1 + O(\varepsilon), \quad \sin\varphi_3 = -1 + O(\varepsilon), \quad \text{if} \quad s_1 = -1$$

$$\sin\varphi_2 = -1 + O(\varepsilon), \quad \sin\varphi_3 = 1 + O(\varepsilon), \quad \text{if} \quad s_2 = 1$$

The degree of instability of the solutions corresponding to the equality $s_1 = 1$ is equal to 2, the solution corresponding to the equality $s_1 = -1$ has a degree of instability equal to 1 and the solution corresponding to the equality $s_2 = 1$ has a degree of instability equal to 2 when $q < A_{\mu\nu}/A_\nu$ and 1 when $q > A_{\mu\nu}/A_\nu$.

It can be shown in a similar way that a solution of the form

$$X \to \infty, \quad Y \to \infty, \quad Y/X \to \infty \quad \text{when} \quad \varepsilon \to 0$$

does not exist for system (3.3).

We introduce the notation

$$d = q/p\nu$$

If $d < 1$, four solutions of the form

$$X \to \infty, \quad Y \to \infty, \quad X/Y \to \infty \quad \text{when} \quad \varepsilon \to 0$$

$$X = \kappa_1\kappa_2 \frac{pqA_\nu A_{\mu\nu}}{\nu}\left(1 - d^2\right)^{1/2}\varepsilon^{-2} + O(\varepsilon^{-1}), \quad Y = \kappa_1 \frac{pA_{\mu\nu}}{\nu}\varepsilon^{-1} + O(1); \quad \kappa_{1,2} = \pm 1$$

$$\cos\varphi_1 = \kappa_1 + O(\varepsilon)$$

$$\cos\varphi_2 = -\kappa_2\left(1 - d^2\right)^{1/2} + O(\varepsilon), \quad \sin\varphi_2 = \kappa_2 d + O(\varepsilon)$$

$$\cos\varphi_3 = \frac{\nu}{qA_\nu}\varepsilon + O(\varepsilon^2), \quad \sin\varphi_3 = -\kappa_2 + O(\varepsilon)$$

exists for system (3.3).

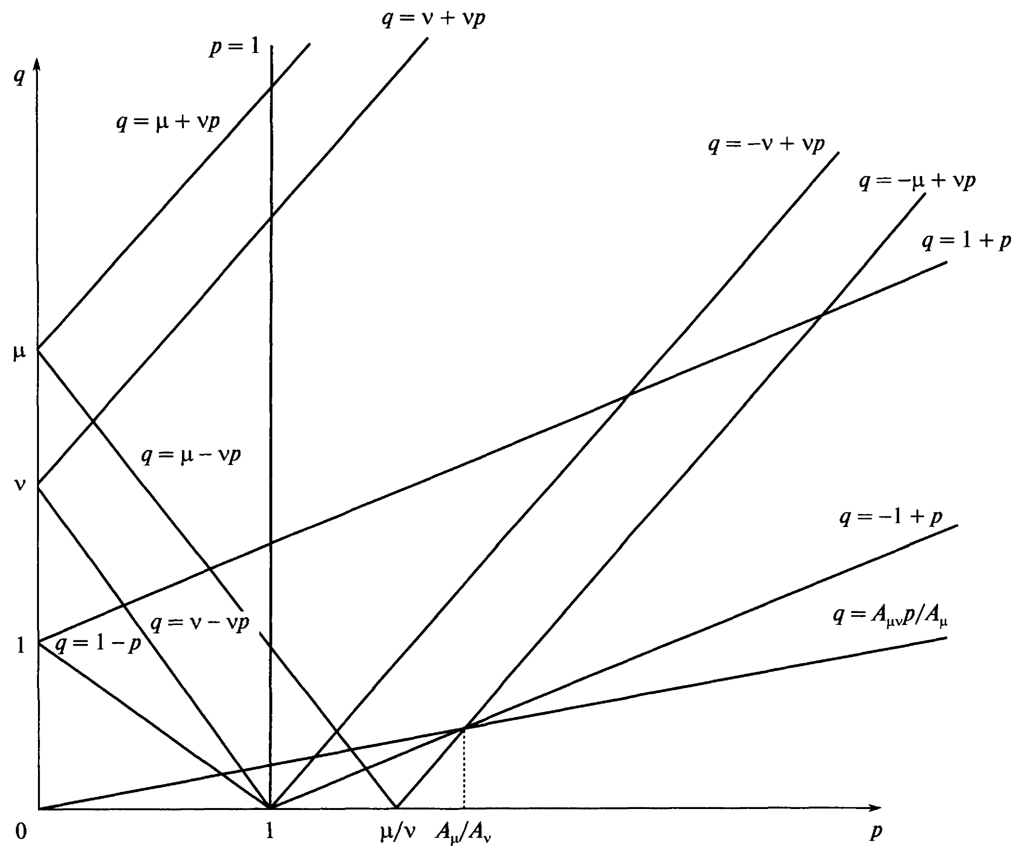

Fig. 6.

The degree of instability of all the solutions is equal to 2.

If $d = 1$, system (3.3) has two asymptotic solutions of the same type

$$X = \kappa_1 p A_\nu A_{\mu\nu}\left(\frac{2p^2\mu}{A_{\mu\nu}}\right)^{1/3}\varepsilon^{-5/3} + O(\varepsilon^{-4/3}),\quad Y = \kappa_1\frac{pA_{\mu\nu}}{\nu}\varepsilon^{-1} + O(\varepsilon^{-2/3})$$

$$\cos\varphi_1 = \kappa_1 + O(\varepsilon)$$

$$\cos\varphi_2 = -\left(\frac{2\mu}{pA_{\mu\nu}}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon^{2/3}),\quad \sin\varphi_2 = 1 + O(\varepsilon^{1/3})$$

$$\cos\varphi_3 = \frac{\varepsilon}{\mathrm{p}\mathrm{A}_\nu} + O(\varepsilon^{4/3}),\quad \sin\varphi_3 = -1 + O(\varepsilon)$$

9. Bifurcation diagram

The analysis of the conditions for asymptotic solutions of system (3.3) to exist presented above enables us to subdivide the plane of the parameters (p, q) into domains with a different number and form of the asymptotic configurations of the pendulum (Fig. 6). The straight lines in Fig. 6 correspond to the boundaries of the domain of existence of some kind of asymptotic solution. It is clear that the number of solutions of the system for fixed inertial characteristics of the system (that is, for fixed μ and ν) is completely defined by its geometric parameters (the relations between the lengths of the links). However, unlike, for example, a double elliptic pendulum, the relations between the masses of the points affect the position of the boundaries (the broken lines) of the domains of existence of the solutions and this also means the number of solutions.

We also note that there are domains in the diagram in which the total number of non-trivial solutions exceeds the number of solutions branching off from the trivial relative equilibria. This is indicative of the fact that, at least for fairly high angular velocities of rotation of the pendulum, relative equilibria of the system exist which do not branch off from trivial equilibrium positions.

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